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Numerical method for the boundary layer problems of non-Newtonian fluid flows along moving surfaces

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. This paper deals with the iterative transformation method for the solution of the boundary layer problem of a non-Newtonian power-law fluid flow along a moving flat surface. Applying similarity transformation to the system of the governing partial differential equations, we derive the boundary value problem of a nonlinear ordinary differential equation on $[0, \infty)$. Numerical solutions are obtained for the velocity components. Moreover, we exhibit the drag coefficient dependence on the velocity ratio and on the power-law exponent.

Keywords: boundary layer problem, non-Newtonian fluid mechanics, similarity solution.

2010 Mathematics Subject Classification: 76A05, 65L10, 35Q35.

1 Introduction

In fluid dynamics, the drag force or force component in the direction of the flow velocity is proportional to the drag coefficient, to the density of the fluid, to the area of the object and the square of the relative speed between the object and the flow velocity. Blasius applied the similarity method to investigate the model arising for a laminar boundary layer of a Newtonian media [3]. Fluids such as molten plastics, pulps, slurries and emulsions, which do not obey the Newtonian law of viscosity, are increasingly produced in the industry. The first analysis of the boundary layer approximations to non-Newtonian media with power-law viscosity was given by Schowalter [13] in 1960. The author derived the equations governing the fluid flow. The numerical solutions to the problem of a laminar flow of the non-Newtonian power-law model past a two-dimensional horizontal surface were presented by Acrivos, Shah and Petersen [1]. When the geometry of the surface is simple the system of differential equations can be examined in details and can be obtained fundamental information about the flow behaviour of

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non-Newtonian fluids in motion (e.g., to predict the drag). It was shown that a non-iterative Töpfer-like transformation can be applied for the determination of the dimensionless wall gradient on a stationary flat surface [4].

Our aim is to examine the drag coefficient in non-Newtonian media along moving flat surfaces applying an iterative method for its numerical evaluation.

2 Mathematical model

Consider an incompressible uniform parallel flow of a non-Newtonian power-law fluid, with a constant velocity U_∞ along an impermeable semi-infinite flat plate whose surface is moving with a constant velocity U_w in the opposite direction to the main stream (see Figure 2.1). The x–axis extends parallel to the plate, while the y-axis extends upwards, normal to it.

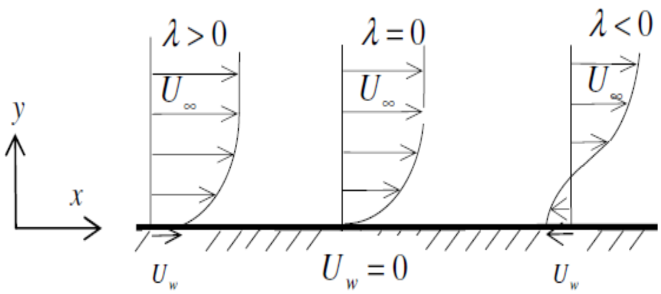

Figure 2.1: Velocity profiles in the boundary layer

Applying the necessary boundary layer approximations, the continuity and momentum equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y}, \tag{2.2}$$

where u, v are the velocity components along x and y coordinates, respectively. The shear stress and the shear rate relation is assumed to be the power-law relation

$$\tau_{yx} = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y},$$

where K stands for the consistency and n is called the power-law index; that is $n < 1$ for pseudoplastic, $n = 1$ for Newtonian, and $n > 1$ for dilatant fluids. Therefore, differential equation (2.2) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu_c\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right), \tag{2.3}$$

$\mu_c = K/\rho$.

For the investigated model, the boundary conditions are formulated such as

$$u|_{y=0} = -U_w, \qquad v|_{y=0} = 0, \qquad u|_{y=+\infty} = U_\infty. \tag{2.4}$$

The continuity equation (2.1) is automatically satisfied by introducing a stream function ψ as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

The momentum equation can be transformed into an ordinary differential equation by the similarity transformations

$$\psi(x,y) = \mu_c^{\frac{1}{n+1}} (U_\infty)^{\frac{2n-1}{n+1}} x^{\frac{1}{n+1}} f(\eta),$$

$$\eta = \mu_c^{-\frac{1}{n+1}} (U_\infty)^{\frac{2-n}{n+1}} y x^{-\frac{1}{n+1}},$$

where η is the similarity variable and $f(\eta)$ is the dimensionless stream function. Equation (2.3) with the transformed boundary conditions can be written as

$$\left(|f''|^{n-1} f''\right)' + \frac{1}{n+1} f f'' = 0, \tag{2.5}$$

$$f(0) = 0, \quad f'(0) = -\lambda, \quad f'(\infty) = \lim_{\eta\to\infty} f'(\eta) = 1, \tag{2.6}$$

where the prime denotes the differentiation with respect to the similarity variable η, and the velocity ratio parameter is

$$\lambda = U_w / U_\infty.$$

Equation (2.5) is called the generalized Blasius equation. It should be noted that for $\lambda > 0$, the fluid and the plate move in the opposite directions, while they move in the same directions if $\lambda < 0$. The dimensionless velocity components have the form:

$$\begin{aligned} u(x,y) &= U_\infty f'(\eta), \\ v(x,y) &= \frac{U_\infty}{n+1} Re_x^{\frac{1}{n+1}} (\eta f'(\eta) - f(\eta)), \end{aligned}$$

and

$$\eta = Re_x^{\frac{1}{n+1}} y/x,$$

where

$$Re_x = U_\infty^{2-n} x^n / \mu_c$$

is the local Reynolds number.

Since the pioneering work by Acrivos [1], different approaches have been investigated for $f''(0) = \gamma$ in the case of non-Newtonian fluids. It has a physical meaning: *drag force* or force due to *skin friction*. It is a fluid dynamic resistive force which is a consequence of the fluid and the pressure distribution on the surface of the object. The *skin friction parameter* γ originates from the non-dimensional *drag coefficient*

$$C_D = (n+1)^{\frac{1}{n+1}} Re^{\frac{-n}{n+1}} |\gamma|^{n-1} \gamma,$$

and it is involved in the *wall shear stress*

$$\tau_w(x) = \left[\frac{\rho^n K U_\infty^{3n}}{x^n}\right]^{\frac{1}{n+1}} |\gamma|^{n-1} \gamma.$$

The boundary value problem (2.5), (2.6) is defined on a semi-infinite interval. For Newtonian fluids ($n = 1$), equation (2.5) is equal to the well-known Blasius equation:

$$f''' + \frac{1}{2} f f'' = 0. \tag{2.7}$$

For non-Newtonian fluids on steady surfaces ($\lambda = 0$), the boundary value problem (2.5), (2.6) has been investigated in [4]. A non-iterative Töpfer-like transformation was introduced for the determination of γ, when

$$f(\eta) = \gamma^{(2n-1)/3} g\left(\gamma^{(2-n)/3}\eta\right)$$

and g is the solution of the initial value problem

$$\left(|g''|^{n-1} g''\right)' + \frac{1}{n+1} g g'' = 0,$$

$$g(0) = 0, \quad g'(0) = 0, \quad g''(0) = 1.$$

By analogy with the Blasius description of Newtonian fluid flows [3], here our aim is to study the similarity solutions and investigate the model arising in the study of a two-dimensional laminar fluid flow with power-law viscosity. A Töpfer-like transformation is applied for the determination of γ.

3 Preliminary results

The existence and uniqueness of Blasius' boundary layer solution to (2.7), (2.6) with $\lambda = 0$ was rigorously proved by Weyl [15]. The properties of similarity solutions to the boundary layer problem on a moving surface ($\lambda \neq 0$) for Newtonian fluids, have been examined by Hussaini and Lakin [7], Hussaini et al. [9]. It turned out that for a semi-infinite plate, the existence of solutions depend on the ratio of the plate surface velocity U_w to the free stream velocity U_∞. When $n = 1$, $\lambda \leq 0$, the existence, uniqueness and analyticity of solution to (2.5), (2.6) were proved by Callegari and Friedman [6] using the Crocco variable formulation. If $\lambda > 0$, Hussaini and Lakin [7] proved that there is a critical value λ_c such that solution exists to (2.7), (2.6) only if $\lambda \leq \lambda_c$ (see [7]). Dual solutions exist for $0 < \lambda < \lambda_c$.

The numerical value of λ_c was found to be 0.3541... The non-uniqueness and analyticity of solution for $\lambda \leq \lambda_c$ has been proved by Hussaini et al. [8, 9].

For non-Newtonian fluids ($n \neq 1$) with $\lambda = 0$, the existence, uniqueness and some analytical results for problem (2.5), (2.6) were established when $0 < n < 1$ by Nachman and Callegari [12]. The existence and uniqueness result for $n > 1$ was considered by Benlahsen et al. [2] via Crocco variable transformation. For non-Newtonian fluids the numerical calculations also show that there is a critical value λ_c for each n such that solution exists only if

$$\lambda \leq \lambda_c$$

(see [10]). The variations of $f''(0)$ and λ_c with λ for different values of n are given via Crocco-like transformation using Runge–Kutta method with shooting technique in [5].

The aim of this paper is to introduce an iterative transformation method for the determination of γ involved in the drag coefficient and the calculation of the boundary layer thickness for different values n and λ.

4 Iterative transformation method

This section is devoted to the application of the scaling concept to numerical analysis of (2.5), (2.6). Solving this problem, we have to deal with a practically unsuited condition at infinity. In the case of $\lambda = 0$ a non-iterative transformation method called Töpfer or Töpfer-like method was be used for solving (2.5), (2.6) either for $n = 1$ [14] or for $n \neq 1$ [5]. Here we apply an iterative transformation method. Non-iterative and iterative transformation methods for boundary value problems have been introduced by Fazio [7].

The idea behind the present method is to consider the "partial" invariance of (2.5), (2.6) with respect to a scaling transformation in the sense that the differential equation and one of the boundary conditions at 0 are invariant, while the other two boundary conditions are not invariant [7]. Therefore, we modify the problem by introducing a numerical parameter h. Now, equation (2.5) is to be solved with boundary conditions

$$f\left(0\right)=0,\quad f'\left(0\right)=-\lambda h,\quad f'\left(\infty\right)=\lim_{\eta\to\infty}f'\left(\eta\right)=1, \tag{4.1}$$

where h is involved, to ensure the invariance of the extended scaling group.

We introduce a Töpfer-like transformation

$$\begin{aligned} g &= \sigma^{\kappa} f, \\ \eta^* &= \sigma^{\mu}\, \eta \end{aligned}$$

to convert the boundary value problem to an initial value problem. Equation (2.5) is scaling invariant if

$$(2-n)\kappa = (1-2n)\mu.$$

Then, one gets

$$\left(\left|g''\right|^{n-1} g''\right)' + \frac{1}{n+1} g g'' = 0. \tag{4.2}$$

Let us choose $\sigma = \gamma$, then

$$g''(0) = \gamma^{\kappa-2\mu+1}$$

and with

$$\kappa - 2\mu + 1 = 0$$

we get

$$\kappa = (1-2n)/3, \qquad \mu = (2-n)/3.$$

Hence, one can obtain the appropriate initial conditions suitable for numerical simulations instead of boundary conditions in the following forms

$$g(0) = 0, \quad g'(0) = -\lambda h^*, \quad g''(0) = 1, \tag{4.3}$$

where h^* is a technical parameter. Moreover, we have

$$h^* = \gamma^{-\frac{n+1}{3}} h \quad \text{and} \quad g'(\infty) = \gamma^{-\frac{n+1}{3}}. \tag{4.4}$$

In order to give the solution to (2.5), (4.1) we have to find a zero of the transformation function $\Gamma\left(h^*\right) = h - 1$. The transformation function and γ are defined implicitly by the solution of the initial value problem (4.2), (4.3).

First, a value of λ is fixed. For given $h^* > 0$,

(i) find g by solving (4.2)–(4.3) numerically with the given h^*

(ii) compute $g'(\infty)$, defined numerically as $g'(\eta^*_\infty)$

(iii) calculate γ from equation $g'(\infty) = \gamma^{-\frac{n+1}{3}}$ (i.e. just take roots)

(iv) define $h := \gamma^{\frac{n+1}{3}} h^*$

(v) let $\Gamma(h^*) := h - 1$.

Then, using this definition, one can indeed apply a root-finder to Γ, such that each step involves the solution of an IVP with g. Using the obtained "numerical root" h^* (for which $\Gamma(h^*) \approx 0$ up to desired tolerance) in condition (4.3), the solution g of the IVP (4.2)–(4.3) must be rescaled to the solution f of (2.5)–(4.1), and since now $h \approx 1$, therefore (4.1) coincides with (2.6) up to the used tolerance, i.e. f solves (2.5)–(2.6) numerically.

Our main interest $f''(0) = \gamma$ involved in the skin friction coefficient is also defined.

From the numerical view point a boundary value problem given on infinite interval has been replaced by an initial value problem involving a technical parameter h into the initial condition.

5 Numerical results

The nonlinear ordinary differential equation (4.2) with initial conditions (4.3) is solved for some values of the power-law index n and velocity ratio parameter λ in the interval $[0, \eta^*_{\max}]$ by MATLAB version R2011a.

An adaptive fourth order Runge–Kutta method was implemented to the initial value problem to find g. A simple secant root-finder is used to define the sequence $\Gamma(h^*_j)$ with a convergence criterion $|\Gamma(h^*_j)| \le 10^{-6}$.

The solution f to problem (2.5), (2.6) is obtained by rescaling from numerical solution for g. The numerical values of $\eta_{\max}$ are also calculated from the numerical solution with condition that $f'(\eta_{\max}) = 0.99$.

The implementation of the secant method is straightforward. The choice of the initial iterates is important. For some positive values of λ, the related numerical results show that $\Gamma(h^*_j)$ has two different zeros and then the boundary value problem has two corresponding solutions (a lower and an upper solution). By setting a λ such that $-1 < \lambda < 0$, $\Gamma(h^*_j)$ has one zero. For the case $\lambda < -1$ we find that Γ has always the same sign and for the root-finder no solution is available. The results for Newtonian flow ($n = 1$) are in agreement with results obtained by Hussaini and Lakin [7].

	λ											
n	-1	-0.8	-0.6	-0.4	-0.2	0	0.15		0.25		0.3	
	one solution						lower	upper	lower	upper	lower	upper
0.5	600	3.754	2.491	2.023	1.807		12.03	1.793	5.531	1.951	3.855	2.169
1	300	4.480	2.977	2.422	2.167		29.84	2.148	9.903	2.315	6.526	2.505
1.5	830	4.960	3.301	2.687	2.407		303.0	4.521	22.66	2.556	10.58	2.738

Table 5.1: The values of h^*

	λ											
n	-1	-0.8	-0.6	-0.4	-0.2	0	0.15		0.25		0.3	
	one solution						lower	upper	lower	upper	lower	upper
0.5	$2.8\ 10^{-6}$	0.071	0.161	0.244	0.306	0.331	0.007	0.0311	0.033	0.262	0.067	0.213
1	$1.2\ 10^{-5}$	0.105	0.195	0.265	0.313	0.332	0.006	0.317	0.032	0.284	0.060	0.252
1.5	$3.1\ 10^{-4}$	0.146	0.239	0.305	0.348	0.365	0.001	0.353	0.024	0.324	0.059	0.298

Table 5.2: The values of $f''(0)$

The results of the numerical calculations are represented for h^*, $f''(0)$, and $\eta_{\max}$ by taking different values for λ and for n in Tables 5.1–5.3. Table 5.1 represents that $f''(0)$ is higher for dilatant fluids ($n > 1$) than for pseudoplastics ($n < 1$).

	λ											
n	-1	-0.8	-0.6	-0.4	-0.2	0	0.15		0.25		0.3	
	one solution						lower	upper	lower	upper	lower	upper
0.5	9001	56.3	37.4	30.3	27.1	26.1	180.44	26.90	82.98	29.28	57.83	32.53
1	300	31.8	25.9	23.3	22.1	21.7	81.94	21.99	47.21	22.82	38.32	23.74
1.5	4.6	6.9	6.4	6.1	5.96	5.92	11.9	4.52	8.21	5.31	7.85	5.99

Table 5.3: The values of $\eta_{\max}$

Figures 5.1–5.3 exhibit the upper and lower solutions for velocities $f' = u(x,y)/U_\infty$ as a function of η for some values of n and λ to show the effect of the velocity parameter λ and power-law exponent n.

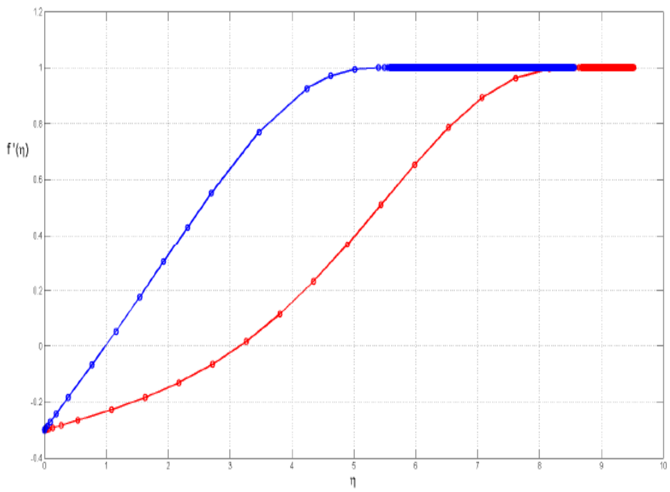

Figure 5.1: Velocity distribution $f'(\eta)$ for $n = 1.5$ and $\lambda = 0.3$

The influences of λ and n on the skin friction parameter γ are represented in Figure 5.4. From the numerical results it is seen that the drag force is reduced for dilatant fluids ($n > 1$) compared to the case $0 < n < 1$. If $\lambda > \lambda_c$, the flow separates. The boundary layer structure collapses and the boundary layer approximations are no longer applicable.

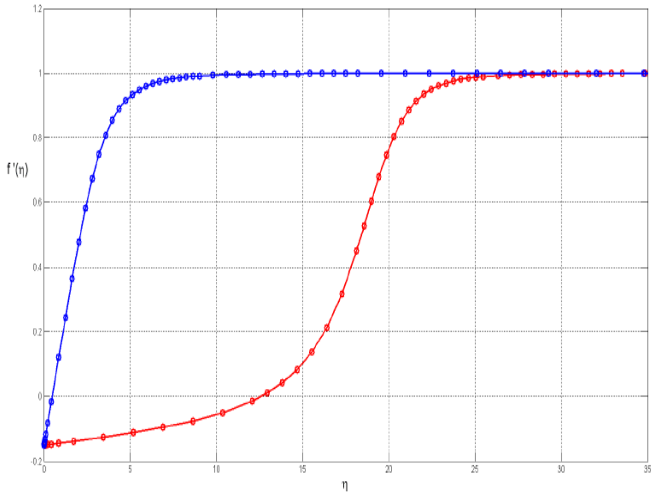

Figure 5.2: Velocity distribution $f'(\eta)$ for $n = 0.5$ and $\lambda = 0.15$

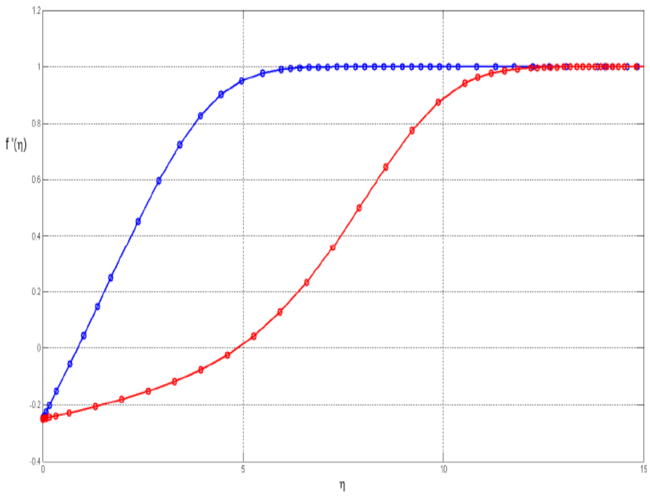

Figure 5.3: Velocity distribution $f'(\eta)$ for $n = 1$ and $\lambda = 0.25$

n	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
λ_c (own)	0.3391	0.3445	0.3495	0.3541	0.3584	0.3624	0.3661	0.3696	0.3728
λ_c [10]		0.3445		0.3541		0.3624		0.3696	

Table 5.4: The values of λ_c

It can be noticed that there are two solutions (upper and lower solutions) for $0 < \lambda < \lambda_c$ (see Figure 5.4). Figures 5.1–5.3 show the effect of the positive parameter λ for different power-law exponent n on γ^n. We remark that f' monotonically increases from $-\lambda$ to 1 for both the lower and upper solutions. This phenomena shows that the velocity component u is monotonically increasing in the boundary layer. Moreover, the boundary layer thickness is higher as n is decreasing. Our results are in good agreement with those reported in [10]. It is shown in Table 5.4, where the critical values of λ_c are compared for our result and for [10].

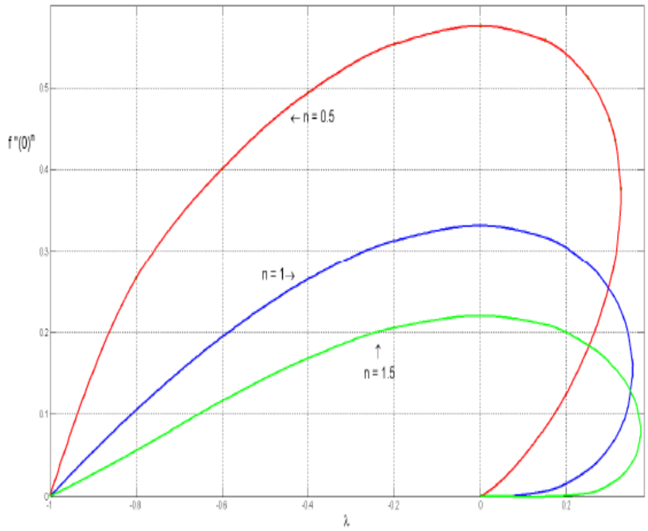

Figure 5.4: The variation of $\gamma^n = [f''(0)]^n$ for different values of power-law index n

The effect of the power exponent n and λ on the profiles for $f''(\eta)$ is exhibited for pseudoplastic, Newtonian and dilatant media on Figures 5.5–5.7. $f''(\eta)$ is included in the shear stress. The boundary layer thickness increases as the value of $\lambda > 0$ increases, and $f''(\eta)$ reaches a maximum in the interior of the flow field. Klemp and Acrivos [11] remarked that at this similarity solution, the downstream influence has not been neglected on the flow. The reason is the lack of the characteristic length in the case of the semi-infinite surface. If the solution exists, it must be self-similar in order to remain independent of whatever length scale is chosen. Therefore, both upstream and downstream effects on the solution at any point in the flow must be such that the shape of the similarity solution.

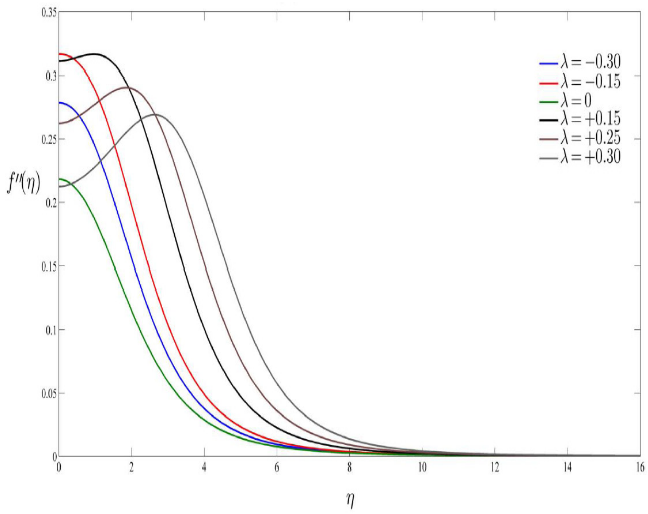

Figure 5.5: The graph of $f''(\eta)$ for $n = 0.5$ applying different λ

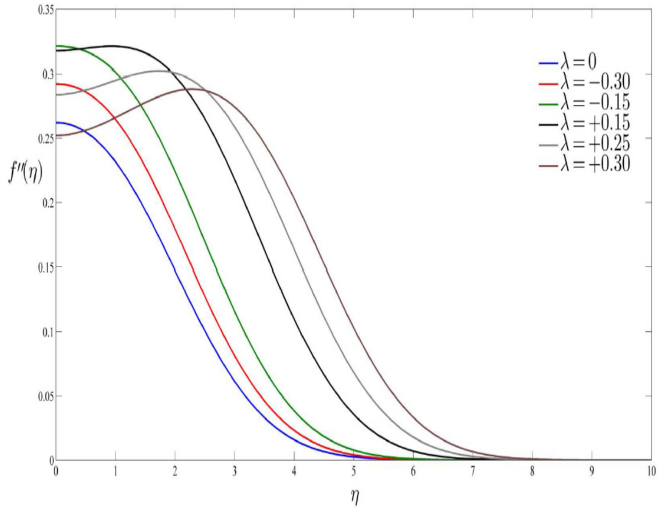

Figure 5.6: The graph of $f''(\eta)$ for $n = 1$ applying different λ

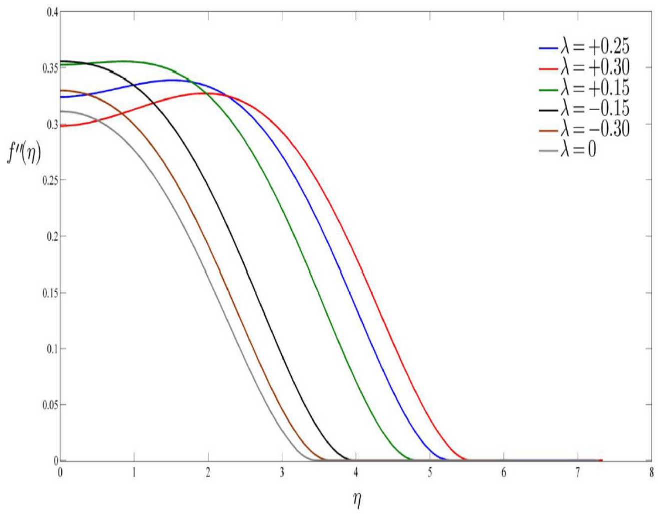

Figure 5.7: The graph of $f''(\eta)$ for $n = 1.5$ applying different λ

References

[1] A. Acrivos, M. J. Shah, E. E. Peterson, Momentum and heat transfer in laminar boundary flow of non-Newtonian fluids past external surfaces, *AIChE J.* **6**(1960), 312–317. url

[2] M. Benlahsen, M. Guedda, R. Kersner, The generalized Blasius equation revisited, *Math. Comput. Modelling* **47**(2008), 1063–1076. MR2413736; url

[3] H. Blasius, Grenzschichten in Flüssigkeiten mit kleiner Reibung (in German), *Z. Math. Phys.* **56**(1908), 1–37.

[4] G. Bognár, Similarity solutions of boundary layer flow for non-Newtonian fluids, *Int. J. Nonlinear Sci. Numer. Simul.* **10**(2009), 1555–1566. url

[5] G. Bognár, On similarity solutions of boundary layer problems with upstream moving wall in non-Newtonian power-law fluids, *IMA J. Appl. Math.* **77**(2012), No. 4, 546–562. MR2957151; url

[6] A. J. Callegari, M. B. Friedman, An analytical solution of a nonlinear, singular boundary value problem in the theory of viscous fluids, *J. Math. Anal. Appl.* **21**(1968), 510–529. MR0224331

[7] R. Fazio, A novel approach to the numerical solution of boundary value problems on infinite intervals, *SIAM J. Numer. Anal.* **33**(1996), 1473–1483. MR1403554; url

[8] M. Y. Hussaini, W. D. Lakin, Existence and nonuniqueness of similarity solutions of a boundary-layer problem, *Quart. J. Mech. Appl. Math.* **39**(1986), 177–191. MR827699; url

[9] M. Y. Hussaini, W. D. Lakin, A. Nachman, On similarity solutions of a boundary layer problem with an upstream moving wall, *SIAM J. Appl. Math.* **47**(1987), 699–709. MR898828; url

[10] A. Ishak, N. Bachok, Power-law fluid flow on a moving wall, *European J. Scie. Res.* **34**(2009), 55–60.

[11] J. B. Klemp, A. A. Acrivos, A method for integrating the boundary-layer equations through a region of reverse flow, *J. Fluid Mech.* **53**(1972), 177–199. url

[12] A. Nachman, A. J. Callegari, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38**(1980), 275–281. MR564014; url

[13] W. R. Schowalter, The application of boundary layer theory to power-law pseudoplastic fluids: Similar solutions, *AIChE J.* **6**(1960), No. 1, 24–28. url

[14] K. Töpfer, Bemerkung zu dem Aufsatz von H. Blasius: Grenzschichten in Flüssigkeiten mit kleiner Reibung (in German), *Z. Math. Phys.* **60**(1912), 397–398.

[15] H. Weyl, On the differential equations of the simplest boundary-layer problems, *Ann. of Math. (2)* **43**(1942), 381–407. MR0006294